Perpetual integral functionals as hitting and occupation times

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Abstract

Let X be a linear diffusion and f a non-negative, Borel measurable function. We are interested in finding conditions on X and f which imply that the perpetual integral functional

$$I_\infty^X(f) := \int_0^\infty f(X_t)\,dt$$

is identical in law with the first hitting time of a point for some other diffusion. This phenomenon may often be explained using random time change. Because of some potential applications in mathematical finance, we are considering mainly the case when X is a Brownian motion with drift $\mu > 0$, denoted $\{B_t^{(\mu)} : \ t \geq 0\}$, but it is obvious that the method presented is more general. We also review the known examples and give new ones. In particular, results concerning one-sided functionals

$$\int_0^\infty f(B_t^{(\mu)})\,\mathbf{1}_{\{B_t^{(\mu)}<0\}}dt \quad \text{and} \quad \int_0^\infty f(B_t^{(\mu)})\,\mathbf{1}_{\{B_t^{(\mu)}>0\}}dt$$

are presented.

This approach generalizes the proof, based on the random time change techniques, of the fact that the Dufresne functional (this corresponds to $f(x) = \exp(-2x)$), playing quite an important rôle in the

study of geometric Brownian motion, is identical in law with the first hitting time for a Bessel process. Another functional arising naturally in this context is

$$\int_0^\infty (a + \exp(B_t^{(\mu)}))^{-2}\, dt,$$

which is seen, in the case $\mu = 1/2$, to be identical in law with the first hitting time for a Brownian motion with drift $\mu = a/2$.

The paper is concluded by discussing how the Feynman-Kac formula can be used to find the distribution of a perpetual integral functional.

Keywords: Time change, Lamperti transformation, Bessel processes, Ray–Knight theorems, Feynman-Kac formula.

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1 Introduction and summary of the results

Let $B^{(\mu)} = \{B_t^{(\mu)} := B_t + \mu t \;:\; t \geq 0\}$ be a Brownian motion with drift $\mu > 0$ and f a non-negative measurable function. Encouraged by a number of examples listed below, we wish to gain better understanding when an integral functional of the type

$$I_\infty(f) := \int_0^\infty f(B_s^{(\mu)})\, ds$$

is identical in law with the first hitting time of a point for some other diffusion. Clearly, we can pose an analogous question for an arbitrary diffusion instead of $B^{(\mu)}$. In fact, the results in Section 2 are fairly easily extended for arbitrary transient diffusions determined by a stochastic differential equation. Our interest in the particular case with $B^{(\mu)}$ is motivated by the numerous studies and results associated to the functional

$$\int_0^\infty \exp(-2B_s^{(\mu)})\, ds. \tag{1}$$

This functional was first considered by Dufresne in [12] where it is seen, among other things, how the functional (1) arises as a perpetuity after a limiting procedure in a discrete model.

We now review some cases of perpetual integral functionals which are identical in law with the first hitting time. Let

$$H_a(Z) := \inf\{t : \; Z_t = a\}$$

denote the first hitting time of the point a for a diffusion Z.

1) In Yor [45] (see [47] for an English translation) it is shown that for the Dufresne functional (1) we have

$$\int_0^\infty \exp(-2B_s^{(\mu)})\, ds \quad \overset{(\mathrm{d})}{=} \quad H_0(R^{(\delta)}), \tag{2}$$

where $R^{(\delta)}$ is a Bessel process of dimension $\delta = 2(1-\mu)$ started at 1, and $\overset{(\mathrm{d})}{=}$ reads "is identical in law with". Recall also that

$$\int_0^\infty \exp(-2B_s^{(\mu)})\, ds \quad \overset{(\mathrm{d})}{=} \quad \frac{1}{2\gamma_\mu}, \tag{3}$$

where γ_μ is a gamma-distributed random variable with parameter μ. We refer to Szabados and Székely [41] for a discussion of Dufresne's functional for random walks.

2) The Ciesielski–Taylor identity:

$$\int_0^\infty \mathbf{1}_{\{R_s^{(\delta+2)}<1\}}\, ds \quad \overset{(\mathrm{d})}{=} \quad H_1(R^{(\delta)}), \tag{4}$$

where $R^{(\delta)}$ and $R^{(\delta+2)}$ are Bessel processes of dimension $\delta > 0$ and $\delta + 2$, respectively, started at 0. For a proof, see Williams [42] p. 159 and 211, and Yor [43], [44] p. 50; in the case $\delta = 1$ there is a pathwise explanation due to D. Williams. We refer also to Getoor and Sharpe [17] p. 98, and to Biane [2] for a generalization to a vast class of pairs of diffusions.

3) The identity due to Biane [2] and Imhof [19]:

$$\int_0^\infty \mathbf{1}_{\{B_s^{(\mu)}<0\}}\, ds \quad \overset{(\mathrm{d})}{=} \quad H_\lambda(B^{(\mu)}), \tag{5}$$

where λ is a random variable independent of $B^{(\mu)}$ and exponentially distributed with parameter 2μ and $B_0^{(\mu)} = 0$.

4) The identity due to Donati-Martin and Yor [10] p. 1044:

$$\int_0^\infty \frac{ds}{\exp(2R_s^{(3)}) - 1} \quad \overset{(\mathrm{d})}{=} \quad H_{\pi/2}(R^{(3)}). \tag{6}$$

where $R^{(3)}$ is a three-dimensional Bessel process started from 0. See [10] also for a probabilistic explanation of (6).

In Section 2 of this paper we present a general method based on Itô's and Tanaka's formulae and random time change techniques which connects the distribution of a perpetual integral functional to the first hitting time. This method gives us the identity (2) and also the following (which could be called the reflecting counterpart of (2)):

$$\int_0^\infty \exp(-2a\, B_s^{(\mu)})\, \mathbf{1}_{\{B_s^{(\mu)}>0\}}\, ds \quad \stackrel{\text{(d)}}{=} \quad H_{1/a}(R^{(\delta)}), \quad \delta = 2\mu/a. \tag{7}$$

where the Bessel process $R^{(\delta)}$ is started at 0 and, in the case $0 < \delta < 2$, reflected at 0. However, the simplest case emerging from our approach leads us to the identity

$$\int_0^\infty (a + \exp(B_s^{(1/2)}))^{-2}\, ds \quad \stackrel{\text{(d)}}{=} \quad H_r(B^{(a/2)}), \tag{8}$$

where $a > 0$, $B_0^{(a/2)} = 0$ and $r = \frac{1}{a}\log(1+a)$. The reflecting counterpart of (8) is

$$\int_0^\infty \frac{\mathbf{1}_{\{B_s^{(1/2)}>0\}}}{(a + \exp(B_s^{(1/2)}))^2}\, ds \quad \stackrel{\text{(d)}}{=} \quad H_r(\tilde{B}^{(a/2)}), \tag{9}$$

where $a > 0$, $\tilde{B}^{(a/2)}$ is reflecting Brownian motion with drift $a/2$ started from 0 and r is as above. In Section 3.3, when analyzing the functional on the left hand side of (8), we also find the diffusion with the first hitting time identical in law with the functional

$$\int_0^\infty (a + \exp(B_s^{(\mu)}))^{-2}\, ds,$$

where $\mu > 0$ is arbitrary. The Laplace transform of this functional can be expressed in terms of Gauss' hypergeometric functions (see [4]).

We have not investigated or constructed a discrete model (as is done in [12] for the functional in (1)) which would lead to the functional in (8). Notice, however, that

$$\int_0^\infty (1 + \exp(B_s^{(\mu)}))^{-2}\, ds = \int_0^\infty \exp(-2\, B_s^{(\mu)})(1 + \exp(-B_s^{(\mu)}))^{-2}\, ds$$

and, hence, this functional can be considered as a modification of Dufresne's functional such that the discounting is bounded (we call this modified functional a translated Dufresne's functional). In fact, using the results in Salminen and Yor [40], where the integrability properties of perpetual functionals are discussed, it is seen that while Dufresne's functional does not have moments of order $m \geq \mu$ (cf. (3)), which is perhaps unrealistic from an economical point of view, the functional in (8) has some exponential moments - being in this respect more appropriate.

For functionals restricted to the negative half line we cannot in general have similar descriptions in terms of first hitting times. A typical example is the identity (5) above. However, the Lamperti transformation allows us to connect exponential functionals to the occupation times for Bessel processes. In Section 3.1 we show the identity

$$\int_0^\infty \exp(-2a\, B_s^{(\mu)})\ \mathbf{1}_{\{B_s^{(\mu)}<0\}}\, ds \quad \stackrel{(\mathrm{d})}{=} \quad \int_0^\infty \mathbf{1}_{\{R_s^{(\delta_2)}>1/a\}}\, ds, \tag{10}$$

where $R^{(\delta_2)}$ is a Bessel process of dimension $\delta_2 = 2(1 - \frac{\mu}{a})$ started at $1/a$. Notice that $R^{(\delta_2)}$ hits 0 in finite time, and, in case $0 < \mu < a$, we take 0 to be a killing boundary point. Further, also by the Lamperti time change, we have

$$\int_0^\infty \exp(2a\, B_s^{(\mu)})\ \mathbf{1}_{\{B_s^{(\mu)}<0\}}\, ds \quad \stackrel{(\mathrm{d})}{=} \quad \int_0^\infty \mathbf{1}_{\{R_s^{(\delta_3)}<1/a\}}\, ds, \tag{11}$$

where $R^{(\delta_3)}$ is a Bessel process of dimension $\delta_3 = 2(1+\frac{\mu}{a})$ started at $1/a$. This identity was first observed and proved in Yor [46] ([47] p. 133) by different methods.

Finally, we recall the recent works by the second author, jointly with H. Matsumoto, see [29], [30], [31], in which the variable

$$\int_0^t \exp(-2B_s^{(\mu)})\, ds, \quad \mu > 0, \tag{12}$$

plays an essential role in obtaining a variant of Pitman's theorem. Indeed, it is proved that the process

$$Z_t^{(\mu)} := \exp(B_t^{(\mu)}) \int_0^t \exp(-2B_s^{(\mu)})\, ds, \quad t > 0,$$

is a time homogeneous diffusion. Other properties of the functional in (12) are studied in the papers [28], [6], [7], [8], [9]. See also Dufresne [13] and Matsumoto and Yor [32].

The paper is organized as follows: In the next section a general method connecting the distribution of a perpetual integral functional to a hitting time is presented. Examples of the method are presented in Section 3. In particular, we discuss the functionals (and their reflected counterparts) appearing in (2) and (8). Further, using the Lamperti transformation we derive the identity (4) from (7) and give a new derivation (for another one, see [39]) for the joint Laplace transform of the functionals

$$\int_0^\infty \frac{\mathbf{1}_{\{B_s^{(1/2)}>0\}}}{(a+\exp(B_s^{(1/2)}))^2}\,ds, \quad L^0_\infty(B^{(1/2)}), \quad \int_0^\infty \frac{\mathbf{1}_{\{B_s^{(1/2)}<0\}}}{(a+\exp(B_s^{(1/2)}))^2}\,ds,$$

where $L^0_\infty(B^{(1/2)})$ is the ultimate value of local time at 0 of $B^{(1/2)}$. Especially, this latter computation has interesting connections to some earlier works. We also prove in Section 3 the identities (5), (46) and (47) (see the table below). In Section 4, we modify the Feynman-Kac formula to be directly applicable for computing the Laplace transform of a perpetual functional and discuss a characterization due to Biane [2] for one-sided functionals. We finish with a short Appendix containing the Ray–Knight theorems used in the paper.

To summarize the discussion made throughout this work, we use systematically random time changes in a set-up which a priori encompasses the scale and speed Feller type representations of one-dimensional diffusions (for which, see, e.g. the recent paper by McKean [33]) as well as Lamperti's transformation and Ray–Knight theorems. Of course, this stochastic approach and the results it allows to derive agree with the more analytic Feynman-Kac approach of solving ODE's, and performing for them the corresponding (deterministic) changes of variables.

We conclude this introduction with a table containing (most of) the functionals and the associated hitting times discussed in this paper. We use the notation $B^{(\mu)}$ for a Brownian motion with drift $\mu > 0$, $R^{(\delta)}$ for a Bessel process of dimension δ, and $\tilde{B}^{(\mu)}$ for a reflected Brownian motion with drift $\mu > 0$. All the processes in the table, if nothing else is said, are started from 0.

Ref.	Functional ($a>0,\ \mu>0$)	Hitting/occupation time
(2)	$\int_0^\infty \exp(-2aB_s^{(\mu)})\,ds$	$H_0(R^{(2-2\mu/a)})$, $R_0^{(2-2\mu/a)}=1/a$
(7)	$\int_0^\infty \exp(-2a\,B_s^{(\mu)})\,\mathbf{1}_{\{B_s^{(\mu)}>0\}}\,ds$	$H_{1/a}(R^{(2\mu/a)})$
(10)	$\int_0^\infty \exp(-2a\,B_s^{(\mu)})\,\mathbf{1}_{\{B_s^{(\mu)}<0\}}\,ds$	$\int_0^\infty \mathbf{1}_{\{R_s^{(2-2\mu/a)}>1/a\}}\,ds$,
(11)	$\int_0^\infty \exp(2a\,B_s^{(\mu)})\,\mathbf{1}_{\{B_s^{(\mu)}<0\}}\,ds$	$\int_0^\infty \mathbf{1}_{\{R_s^{(2+2\mu/a)}<1/a\}}\,ds$, $R_0^{(2+2\mu/a)}=1/a$
(8)	$\int_0^\infty (a+\exp(B_s^{(1/2)}))^{-2}\,ds$	$H_r(B^{(a/2)})$, $r=\frac{1}{a}\log(1+a)$
(5)	$\int_0^\infty \mathbf{1}_{\{B_s^{(\mu)}<0\}}\,ds$	$H_\lambda(B^{(\mu)}),\quad \lambda\sim \mathrm{Exp}(2\mu)$
(4)	$\int_0^\infty \mathbf{1}_{\{R_s^{(\delta+2)}<1\}}\,ds$	$H_1(R^{(\delta)}),\ \delta>0$
(6)	$\int_0^\infty (\exp(2R_s^{(3)})-1)^{-1}\,ds$	$H_{\pi/2}(R^{(3)})$
(46)	$\int_0^\infty \exp(-2R_s^{(3)})\,ds$	$H_1(R^{(2)})$
(47)	$\int_0^\infty (1+R_s^{(3)})^{-2\gamma}\,ds,\quad \gamma>1$	$H_{1/(\gamma-1)}(R^{((2\gamma-1)/(\gamma-1))})$

2 Perpetual integral functionals as first hitting times

Let $\{B_t^{(\mu)}:\ t\geq 0\}$ be a Brownian motion with drift $\mu>0$ and f a non-negative locally integrable function. Then (see Engelbert and Senf [16] and Salminen and Yor [40])

$$I_\infty(f):=\int_0^\infty f(B_u^{(\mu)})\,du<\infty\quad\text{a.s.}\quad\Leftrightarrow\int^\infty f(x)\,dx<\infty. \tag{13}$$

In our first proposition it is stated, under some additional assumptions on f, that there exists a diffusion constructed in the same probability space as $B^{(\mu)}$ such that the perpetual integral functional $I_\infty(f)$ is a.s. equal to the first hitting time of a point for this diffusion.

Proposition 2.1 *Let $f:\mathbf{R}\mapsto\mathbf{R}$ be a monotone, twice continuously differentiable function such that $r:=\lim_{x\to\infty}f(x)$ exists. Introduce the additive functional*

$$I_s:=\int_0^s(f'(B_u^{(\mu)}))^2\,du.$$

Assume further that $f'(x)\neq 0$ and

$$\int^\infty(f'(x))^2dx<\infty.$$

Let Z be a diffusion given by

$$Z_t:=f(B_{\alpha_t}^{(\mu)}),$$

for $t\geq 0$ such that

$$\alpha_t:=\inf\{s:\ I_s>t\}<\infty.$$

Then a.s.

$$\int_0^\infty(f'(B_s^{(\mu)}))^2\,ds=\inf\{t:\ Z_t=r\}. \tag{14}$$

Moreover, Z is a solution of the SDE

$$dZ_t=d\beta_t+G\circ f^{-1}(Z_t)\,dt,\quad Z_0=f(0),$$

where β is a Brownian motion and

$$G(x):=\frac{1}{(f'(x))^2}\Big(\frac{1}{2}f''(x)+\mu\,f'(x)\Big). \tag{15}$$

Proof To fix ideas, take f to be increasing. By Ito's formula

$$\begin{aligned} f(B_u^{(\mu)}) - f(0) &= \int_0^u f'(B_s^{(\mu)})\, dB_s^{(\mu)} + \frac{1}{2}\int_0^u f''(B_s^{(\mu)})\, ds \\ &= \int_0^u f'(B_s^{(\mu)})\, dB_s + \int_0^u (f'(B_s^{(\mu)}))^2\, G(B_s^{(\mu)})\, ds. \end{aligned}$$

Replacing u by α_t we obtain

$$Z_t - Z_0 = \int_0^{\alpha_t} f'(B_s^{(\mu)})\, dB_s + \int_0^{\alpha_t} (f'(B_s^{(\mu)}))^2\, G(B_s^{(\mu)})\, ds.$$

Because

$$I'_s = (f'(B_s^{(\mu)}))^2 \quad \text{and} \quad \alpha'_t = \frac{1}{I'_{\alpha_t}} = (f'(B_{\alpha_t}^{(\mu)}))^{-2}$$

it follows from Lévy's theorem that

$$\beta_t := \int_0^{\alpha_t} f'(B_s^{(\mu)})\, dB_s,\ t \geq 0,$$

is a Brownian motion, and we obtain the claimed SDE:

$$\begin{aligned} Z_t - Z_0 &= \beta_t + \int_0^t (f'(B_{\alpha_s}^{(\mu)}))^2\, G(B_{\alpha_s}^{(\mu)})\, d\alpha_s \\ &= \beta_t + \int_0^t G \circ f^{-1}(Z_s)\, ds. \end{aligned}$$

Because $B_t^{(\mu)} \to \infty$ as $t \to \infty$ we have

$$\lim_{t\to\infty} f(B_t^{(\mu)}) = r, \quad \text{a.s.}$$

It follows now from

$$Z_{I_t} = f(B_t^{(\mu)}) < r$$

by letting $t \to \infty$ that

$$I_\infty \;=\; H_r(Z) \quad \text{a.s.},$$

completing the proof. □

Remark 2.2 Let f and Z be as in Proposition 2.1. Then, modifying the proof above, it is seen for $x>0$ that

$$\int_0^{H_x(B^{(\mu)})} (f'(B_s^{(\mu)}))^2 \, ds = \inf\{t : \ Z_t = f(x)\} \quad \text{a.s.} \tag{16}$$

We consider next perpetual integral functionals restricted on $\mathbf{R}_+$. For simplicity we take f to be decreasing, and leave the case "f increasing" to the reader. Further, we use the notation

$$M_t^{(-\mu)} := \sup\{B_s^{(-\mu)} : \ s \le t\} := \sup\{B_s - \mu\, s : \ s \le t\}, \ \mu > 0.$$

Recall that

$$\{\rho_t^{(\mu)} := M_t^{(-\mu)} - B_t^{(-\mu)} : \ t \ge 0\} \tag{17}$$

is identical in law to a reflecting Brownian motion with drift μ, i.e., the solution of Skorokhod's reflection equation driven by

$$-B_t^{(-\mu)} = (-B_t) + \mu\, t, \quad t \ge 0.$$

Proposition 2.3 *Let f and G be as in Proposition 2.1. Assume moreover that f is decreasing and let $\rho^{(\mu)}$ be as given in (17). Let*

$$A_t^+ := \int_0^t (f'(\rho_s^{(\mu)}))^2 \, ds, \quad \text{and} \quad \alpha_t^+ := \inf\{s : \ A_s^+ > t\}.$$

Define a process Z via

$$Z_t := f(0) - f(\rho^{(\mu)}_{\alpha_t^+})$$

for $t \ge 0$ such that $\alpha_t^+ < \infty$. Then

$$I_\infty^+ := \int_0^\infty (f'(B_s^{(\mu)}))^2 \, \mathbf{1}_{\{B_s^{(\mu)} > 0\}} \, ds \overset{(\mathrm{d})}{=} \inf\{t : \ Z_t = r^\star\}, \tag{18}$$

where $r^\star := f(0) - r$. The process Z is a solution of the reflected SDE

$$dZ_t = d\beta_t - G \circ f^{-1}(f(0) - Z_t)\, dt + L_t^0(Z), \quad Z_0 = 0,$$

where β is a Brownian motion, $\{L_t^0(Z) : \ t \ge 0\}$ is a non-decreasing process which increases only on the zero set of Z, and G is as in Proposition 2.1.

Remark 2.4 In comparison with the statement (14) in Proposition 2.1, notice that the equality in (18) holds "only" in distribution.

Proof The starting point is the observation

$$I_\infty^+ \stackrel{(\mathrm{d})}{=} A_\infty^+,$$

and, hence, we find the law of A_∞^+. As in the proof of Proposition 2.1 we use Itô's formula

$$\begin{aligned} f(0) &- f(\rho_t^{(\mu)}) \\ &= -\int_0^t f'(\rho_s^{(\mu)})\,(dM_s^{(-\mu)} - dB_s^{(-\mu)}) - \frac{1}{2}\int_0^t f''(\rho_s^{(\mu)})\,ds \\ &= \int_0^t f'(\rho_s^{(\mu)})\,dB_s - f'(0)M_t^{(-\mu)} - \int_0^t \Big(\frac{1}{2}f''(\rho_s^{(\mu)}) + \mu f'(\rho_s^{(\mu)})\Big)\,ds. \end{aligned}$$

Consequently, for $t \geq 0$ such that $\alpha_t^+ < \infty$

$$\begin{aligned} Z_t &= \int_0^{\alpha_t^+} f'(\rho_s^{(\mu)})\,dB_s - f'(0)M_{\alpha_t^+}^{(-\mu)} - \int_0^{\alpha_t^+} (f'(\rho_s^{(\mu)}))^2\; G(\rho_s^{(\mu)})\,ds \\ &= \beta_t - \int_0^t G\circ f^{-1}(f(0) - Z_s)\,ds + L_t^0(Z), \end{aligned}$$

where $L_t^0(Z) := -f'(0)M_{\alpha_t^+}^{(-\mu)}$ determines a non-decreasing process which increases only on the zero set of $\{Z_t : \; t \geq 0\}$. It holds also that

$$0 \leq Z_t < r^* := f(0) - r$$

for all $t \geq 0$ such that $\alpha_t^+ < \infty$. To conclude the proof, observe that

$$Z_{A_\infty^+} = \lim_{t\to\infty}\Big(f(0) - f(\rho_t^{(\mu)})\Big) = r^*$$

giving (18). □

Remark 2.5 The method of random time change is also used by Getoor and Sharpe [17] Section 5 as a tool to compute the distributions of stopped functionals of Brownian motion. Their idea is similar to the one in Propositions 2.1 and 2.3, that is, to identify the functional with the first hitting time of a point for some other diffusion.

3 Detailed studies of some perpetual functionals

3.1 Dufresne's functional

As a very particular case of our method, we study now the identity (2) (or (19) below) and its various one-sided variants. For more details concerning (2), we refer to [47] p. 16. For the joint distribution of the functionals in (21) and (22), see Salminen and Yor [39]. In Section 4 Example 4.7 an additional characterization is derived for the functionals in (22).

Proposition 3.1 *Let $B^{(\mu)}$ with $\mu > 0$ be started from 0. Then for $a > 0$ the following 5 identities hold*
a: (cf. (2)),

$$\int_0^\infty \exp(-2aB_s^{(\mu)})\, ds \quad \stackrel{(d)}{=} \quad H_0(R^{(2-2\mu/a)}), \tag{19}$$

where $R^{(2-2\mu/a)}$ is a Bessel process of dimension $2-2\mu/a$ started at $1/a$.
b: (cf. (7)),

$$\int_0^\infty \exp(-2a\, B_s^{(\mu)})\, \mathbf{1}_{\{B_s^{(\mu)}>0\}}\, ds \quad \stackrel{(d)}{=} \quad H_{1/a}(R^{(2\mu/a)}), \tag{20}$$

where $R^{(2\mu/a)}$ is started at 0 and, in the case $0 < \mu < a$, reflected at 0.
c: (cf. (7)),

$$\int_0^\infty \exp(-2a\, B_s^{(\mu)})\, \mathbf{1}_{\{B_s^{(\mu)}>0\}}\, ds \quad \stackrel{(d)}{=} \quad \int_0^\infty \mathbf{1}_{\{R_s^{(2-2\mu/a)}<1/a\}}\, ds, \tag{21}$$

where $R^{(2-2\mu/a)}$ is started at $1/a$ and killed when it hits 0.
d: (cf. (10) and (69)),

$$\int_0^\infty \exp(-2a\, B_s^{(\mu)})\, \mathbf{1}_{\{B_s^{(\mu)}<0\}}\, ds \quad \stackrel{(d)}{=} \quad \int_0^\infty \mathbf{1}_{\{R_s^{(2-2\mu/a)}>1/a\}}\, ds, \tag{22}$$

where $R^{(2-2\mu/a)}$ is started from $1/a$ and killed when it hits 0.
e: (cf. (11)),

$$\int_0^\infty \exp(2a\, B_s^{(\mu)})\, \mathbf{1}_{\{B_s^{(\mu)}<0\}}\, ds \quad \stackrel{(d)}{=} \quad \int_0^\infty \mathbf{1}_{\{R_s^{(2+2\mu/a)}<1/a\}}\, ds, \tag{23}$$

where $R_0^{(2+2\mu/a)} = 1/a$.

Remark 3.2 In fact, as is clear from the proofs below, the identities (19), (21), (22), and (23) hold a.s., i.e., the Bessel processes appearing therein can be constructed in the same probability space where $B^{(\mu)}$ is given.

Proof

a: We let $a = 1$; the case with arbitrary a can be treated using the scaling property (however, see the proof of (20) where we work with arbitrary a). Apply Proposition 2.1 with $f(x) = \exp(-x)$. Now f is decreasing, and straightforward computations show that

$$G(x) = (\frac{1}{2} - \mu) \exp(x).$$

The process Z is given by the SDE

$$d\, Z_t = d\, \beta_t + \frac{1 - 2\mu}{2\, Z_t}\, dt, \quad Z_0 = f(0) = 1.$$

Consequently, Z is a Bessel process of dimension $\delta = 2 - 2\mu$, as claimed in (19).

b: Here we use Proposition 2.3 with $f(x) = a^{-1} \exp(-ax)$ which leads us to consider the additive functional

$$A_t^+ := \int_0^t \exp(-2a\, \rho_s^{(\mu)})\, ds$$

where $\rho^{(\mu)}$ is a reflecting Brownian motion with drift $\mu > 0$ (see (17)). By Proposition 2.3 the process Z given by

$$Z_t := \frac{1}{a}(1 - \exp(-a\, \rho^{(\mu)}_{\alpha_t^+}))$$

satisfies the SDE

$$Z_t = \beta_t + (\frac{\mu}{a} - \frac{1}{2}) \int_0^t (\frac{1}{a} - Z_s)^{-1}\, ds + L_t^0(Z),\ Z_0 = 0.$$

Recall that α^+ is the inverse of A^+, and $L^0(Z)$ is a non-decreasing process which increases only on the zero set of Z. Proposition 2.3 yields now

$$\begin{aligned} \int_0^\infty \exp(-2a\, B_s^{(\mu)})\, \mathbf{1}_{\{B_s^{(\mu)} > 0\}}\, ds \quad &\stackrel{\text{(d)}}{=} \quad \int_0^\infty \exp(-2a\, \rho_s^{(\mu)})\, ds \\ &\stackrel{\text{(d)}}{=} \quad H_{1/a}(Z). \end{aligned}$$

It remains to show (cf. (20)) that $H_{1/a}(Z)$ is identical in law with $H_{1/a}(R^{(2\mu/a)})$. One way to do this is to compute the Laplace transform of $H_{1/a}(Z)$ by using the standard diffusion theory (see Itô and McKean [20] or [3] II.10 p. 18). A slightly shorter way is to observe that $\{1/a - Z_t :\ t \geq 0\}$ is a Bessel process of dimension $2-2\mu/a$ started at $1/a$, reflected at $1/a$ and killed when it hits 0, denoted $\tilde{R}^{(2-2\mu/a)}$. This gives us

$$H_{1/a}(Z) \quad \stackrel{\text{(d)}}{=} \quad H_0(\tilde{R}^{(2-2\mu/a)}).$$

Finally, the claim

$$H_0(\tilde{R}^{(2-2\mu/a)}) \quad \stackrel{\text{(d)}}{=} \quad H_{1/a}(R^{(2\mu/a)}), \tag{24}$$

where the Bessel process $R^{(2\mu/a)}$ is started at 0, can be verified by straightforward but lenghty computations with Laplace transforms. (As seen in Section 3.2, the Ciesielski–Taylor identity follows easily from (24)).

c & d: To prove (21) and (22) we use the Lamperti transformation (see Lamperti [23]) as in the proof of Dufresne's identity in Yor [45]. Notice that the Lamperti transformation is used (and proved) implicitly also in the above proof of (19). To recall the transformation, let

$$A_t^{(1)} = \int_0^t \exp(-2a\, B_s^{(\mu)})\, ds$$

and

$$\alpha_t^{(1)} := \inf\{s :\ A_s^{(1)} > t\}.$$

Then the (first) Lamperti transformation says that the process

$$\{\frac{1}{a} \exp(-a\, B^{(\mu)}_{\alpha_t^{(1)}}) :\ t \geq 0\} \tag{25}$$

is a Bessel process of dimension $2 - 2\mu/a$ started at $1/a$ and killed when it hits 0. Letting $R^{(2-2\mu/a)}$ denote this Bessel process we obtain

$$\int_0^\infty \exp(-2a\, B_s^{(\mu)})\, \mathbf{1}_{\{B_s^{(\mu)}>0\}}\, ds$$
$$= \int_0^\infty \exp(-2a\, B^{(\mu)}_{\alpha_s^{(1)}})\, \mathbf{1}_{\{B^{(\mu)}_{\alpha_s^{(1)}}>0\}}\, d\alpha_s^{(1)}$$

$$
\begin{aligned}
&= \int_0^{H_0(R^{(2-2\mu/a)})} \mathbf{1}_{\{R_s^{(2-2\mu/a)}<1/a\}}\, ds && (26)\\
&= \int_0^{\infty} \mathbf{1}_{\{R_s^{(2-2\mu/a)}<1/a\}}\, ds.
\end{aligned}
$$

For (22) we have similarly

$$
\begin{aligned}
&\int_0^{\infty} \exp(-2a\, B_s^{(\mu)})\, \mathbf{1}_{\{B_s^{(\mu)}<0\}}\, ds \\
&\qquad = \int_0^{\infty} \exp(-2a\, B^{(\mu)}_{\alpha_s^{(1)}})\, \mathbf{1}_{\{B^{(\mu)}_{\alpha_s^{(1)}}<0\}}\, d\alpha_s^{(1)} \\
&\qquad = \int_0^{H_0(R^{(2-2\mu/a)})} \mathbf{1}_{\{R_s^{(2-2\mu/a)}>1/a\}}\, ds. && (27)\\
&\qquad = \int_0^{\infty} \mathbf{1}_{\{R_s^{(2-2\mu/a)}>1/a\}}\, ds.
\end{aligned}
$$

e: The identity (23) can be proved using the Lamperti transformation (25) with negative μ. This can also be formulated for $\mu > 0$ by defining

$$
A_t^{(2)} := \int_0^t \exp(2a\, B_s^{(\mu)})\, ds
$$

and

$$
\alpha_t^{(2)} := \inf\{s :\ A_s^{(2)} > t\}.
$$

Then the (second) Lamperti transformation states that

$$
\{\frac{1}{a} \exp(a\, B^{(\mu)}_{\alpha_t^{(2)}}) :\ t \ge 0\} \tag{28}
$$

is a Bessel process of dimension $2 + 2\mu/a$, denoted $R^{(2+2\mu/a)}$, started at $1/a$. Consequently, as in the proofs of **c** and **d** above,

$$
\int_0^{\infty} \exp(2a\, B_s^{(\mu)})\ \mathbf{1}_{\{B_s^{(\mu)}<0\}}\, ds = \int_0^{\infty} \mathbf{1}_{\{R_s^{(2+2\mu/a)}<1/a\}}\, ds.
$$

□

Remark 3.3 **(1)** Notice that in case $a = 1$ and $\mu = 1/2$ the identity (20) (see also (26)) takes the form

$$\begin{aligned}\int_0^\infty \exp(-2\,B_s^{(1/2)})\,\mathbf{1}_{\{B_s^{(1/2)}>0\}}\,ds &\stackrel{\text{(d)}}{=} H_1(\rho)\\ &\stackrel{\text{(d)}}{=} \int_0^{H_0(B)} \mathbf{1}_{\{B_s<1\}}\,ds\end{aligned}$$

where ρ is a reflecting Brownian motion started at 0. Similarly, (27) can be written as

$$\int_0^\infty \exp(-2\,B_s^{(1/2)})\,\mathbf{1}_{\{B_s^{(1/2)}<0\}}\,ds = \int_0^{H_0(B)} \mathbf{1}_{\{B_s>1\}}\,ds.$$

From [3] formula 1.2.4.1 p. 200, for $\gamma \geq 0$

$$\mathbf{E}_1\Big(\exp\Big(-\gamma\int_0^{H_0(B)} \mathbf{1}_{\{B_s>1\}}\,ds\Big)\Big) = \frac{1}{1+\sqrt{2\gamma}}.$$

By the well known formula (cf. [3] formula 1.2.0.1 p. 198)

$$\mathbf{E}_x\Big(\exp(-\gamma H_0(B))\Big) = \exp(-x\sqrt{2\gamma}), \quad x > 0,\ \gamma \geq 0,$$

and, therefore,

$$\int_0^\infty \exp(-2\,B_s^{(1/2)})\,\mathbf{1}_{\{B_s^{(1/2)}<0\}}\,ds \stackrel{\text{(d)}}{=} H_\lambda(B),$$

where B is started at 0 and λ is an exponentially (with parameter 1) distributed random variable independent of B.

(2) By the Lamperti time changes (25) and (28), a general perpetual integral functional of geometric Brownian motion can be expressed in terms of Bessel processes as follows:

$$\begin{aligned}\int_0^\infty f(\exp(a\,B_s^{(\mu)}))\,ds &= \int_0^\infty (a\,R_s^{(2+2\mu/a)})^{-2}\ f(a\,R_s^{(2+2\mu/a)})\ ds\\ &= \int_0^\infty (a\,R_s^{(2-2\mu/a)})^{-2}\ f\Big(\frac{1}{a\,R_s^{(2-2\mu/a)}}\Big)\ ds,\end{aligned}$$

where $R^{(2-2\mu/a)}$ is killed when it hits 0.

3.2 Ciesielski–Taylor identity

The identity (20) (cf. also (24)) is now applied to deduce the well known and puzzling Ciesielski–Taylor identity (4) (or (29) below), see Ciesielski and Taylor [5], and also Yor [43] and [44], Chap. IV.

Proposition 3.4 *The following identity in law holds:*

$$\int_0^\infty \mathbf{1}_{\{R_s^{(\delta+2)}<1/a\}}\, ds \quad \overset{\text{(d)}}{=} \quad H_{1/a}(R^{(\delta)}), \quad \forall\, \delta > 0 \tag{29}$$

where the Bessel processes are started at 0.

Proof By Proposition 3.1 **b** and **c**

$$H_{1/a}(R^{(2\mu/a)}) \quad \overset{\text{(d)}}{=} \quad \int_0^\infty \mathbf{1}_{\{R_t^{(2-2\mu/a)}<1/a\}}\, dt,$$

where $R^{(2-2\mu/a)}$ is started from $1/a$. Using the standard time reversal argument (see Pitman and Yor [34] p. 341) we obtain

$$\int_0^\infty \mathbf{1}_{\{R_t^{(2-2\mu/a)}<1/a\}}\, dt \quad \overset{\text{(d)}}{=} \quad \int_0^\infty \mathbf{1}_{\{R_t^{(2+2\mu/a)}<1/a\}}\, dt,$$

where $R^{(2+2\mu/a)}$ is started at 0. Letting $\delta = 2\mu/a$ gives now the claimed identity (29). □

3.3 Translated Dufresne's functional

An interesting case emerging from Proposition 2.1 is when the function G in (15) is equal to a constant c, say. Straightforward computations show that this yields us the functional

$$\int_0^\infty (\frac{c}{\mu} + A\exp(2\mu B_s^{(\mu)}))^{-2}\, ds,$$

where A is a free constant. It is natural to assume that $c > 0$ and $A > 0$. By the scaling property of Brownian motion,

$$\int_0^\infty (\frac{c}{\mu} + A\exp(2\mu B_s^{(\mu)}))^{-2}\, ds \overset{\text{(d)}}{=} \frac{1}{4\mu^2}\int_0^\infty (\frac{c}{\mu} + A\exp(B_s^{(1/2)}))^{-2}\, ds$$

indicating that we are, in fact, obtaining information only about Brownian motion with drift $\mu = 1/2$. Choosing the constants appropriately, this functional, which in [39] is called Dufresne's translated perpetuity, can be expressed in the form

$$\int_0^\infty (a\ \exp(B_s^{(1/2)}) + 1)^{-2}\, ds$$

where $a > 0$. In [39] we derive the joint Laplace transform for the one sided variants of this functional, that is, for the functionals in (31) and (32) below. Here we give a new derivation of this Laplace transform based on the Ray–Knight theorem for Brownian motion stopped at the first hitting time. However, before this, we characterize the functional and its one-sided variants via hitting times.

Proposition 3.5 *Suppose that $B_0^{(1/2)} = 0$. Then the following 3 identities hold*

a:

$$\int_0^\infty (a \exp(B_s^{(1/2)}) + 1)^{-2}\, ds \quad \overset{\text{(d)}}{=} \quad H_r(\beta^{(1/2)}), \tag{30}$$

where $a > 0$, $r = \log((1+a)/a)$, and $\beta^{(1/2)}$ is a Brownian motion with drift $1/2$ started at 0.

b:

$$\int_0^\infty \Big(a \exp(B_s^{(1/2)}) + 1\Big)^{-2} \mathbf{1}_{\{B_s^{(1/2)} > 0\}} ds \quad \overset{\text{(d)}}{=} \quad H_r(\tilde\beta^{(1/2)}), \tag{31}$$

where a and r are as above, and $\tilde\beta^{(1/2)}$ is reflecting Brownian motion with drift $1/2$ started at 0.

c:

$$\int_0^\infty \Big(a \exp(B_s^{(1/2)}) + 1\Big)^{-2} \mathbf{1}_{\{B_s^{(1/2)} < 0\}} ds \overset{\text{(d)}}{=} H_\lambda(\beta^{(1/2)}), \tag{32}$$

where $a > -1$, λ is exponentially distributed with parameter $(1 + a)$, and $\beta^{(1/2)}$ is as in (30).

Proof

a: Instead of simply refering to Proposition 2.1, it is perhaps more instructive to go through the computation; in this way we also gain better understanding why the case $\mu = 1/2$ is a special one. Define

$$f(x) := \log\Big(\frac{a}{a + \exp(-x)}\Big),$$

and observe that

$$f'(x) = (a\exp(x)+1)^{-1}.$$

By Itô's formula

$$\log\left(a+\exp(-B_t^{(\mu)})\right) - \log(a+1) = -\int_0^t \frac{dB_s}{a\exp(B_s^{(\mu)})+1}$$

$$+(\frac{1}{2}-\mu)\int_0^t \frac{ds}{a\exp(B_s^{(\mu)})+1} - \frac{1}{2}\int_0^t \frac{ds}{(a\exp(B_s^{(\mu)})+1)^2}. \tag{33}$$

Introduce

$$A_t := \int_0^t \frac{ds}{(a\exp(B_s^{(\mu)})+1)^2},$$

and

$$Z_u := f(B_{\alpha_u}^{(\mu)}) = \log\Big(\frac{a}{a+\exp(-B_{\alpha_u}^{(\mu)})}\Big),$$

where α is the inverse of A. We have $Z_0 = \log(a/(1+a)) < 0$, and $Z_u < 0$ for all u such that $\alpha_u < \infty$. Moreover, from (33) it is seen that Z is a solution of the SDE

$$Z_u = Z_0 + \beta_u^{(1/2)} + (\mu - \frac{1}{2})\int_0^u \frac{ds}{1-\exp(Z_s)},$$

where $\beta^{(1/2)}$ is a Brownian motion with drift $1/2$ starting from 0. Consequently, because $Z_u \to 0$ as $u \to \infty$ we obtain, as explained in the proof of Proposition 2.1,

$$\int_0^\infty \frac{ds}{(a\exp(B_s^{(\mu)})+1)^2} = H_0(Z),$$

and, in particular, for $\mu = 1/2$ we have (30).

b: The identity (31) can be proved by directly applying Proposition 2.3, and we leave the details to the reader.

c: Consider now the identity (32). Our proof of this relies on the Ray–Knight Theorem 5.2 given in Appendix at the end of the paper. Firstly, for arbitrary $\mu > 0$ we have by the occupation time formula

$$\int_0^\infty \Big(a\exp(B_s^{(\mu)})+1\Big)^{-2}\mathbf{1}_{\{B_s^{(\mu)}<0\}}ds = \int_{-\infty}^0 \Big(a\exp(y)+1\Big)^{-2} L_\infty^y(B^{(\mu)})\,dy,$$

where $L_\infty^y(B^{(\mu)})$ is the total local time of $B^{(\mu)}$ at level y. By Theorem 5.2,

$$\int_{-\infty}^0 \Big(a\exp(y)+1\Big)^{-2} L_\infty^y(B^{(\mu)})\,dy, \quad \stackrel{\text{(d)}}{=} \quad \int_0^\infty \Big(a\exp(-y)+1\Big)^{-2} Z_y\,dy,$$

where Z satisfies the SDE

$$dZ_y = 2\sqrt{Z_y}\, dB_y - 2\mu Z_y\, dy.$$

The distribution of Z_0 is the distribution of $L^0_\infty(B^{(\mu)})$, i.e., the exponential distribution with parameter μ. Consider

$$\begin{aligned}
(a\ \exp(-y)+1)^{-1}\, Z_y &- (a+1)^{-1}\, Z_0 \\
&= 2\int_0^y (a\ \exp(-u)+1)^{-1}\sqrt{Z_u}\, dB_u \\
&\qquad - \int_0^y \frac{2\mu + (2\mu-1)\, a\ \exp(-u)}{(a\ \exp(-u)+1)^{-2}}\, Z_u\, du. \qquad (34)
\end{aligned}$$

Define

$$C_y := \int_0^y (a\ \exp(-u)+1)^{-2}\, Z_u\, du, \quad y \geq 0,$$

and let c denote the inverse of C. From (34) we obtain for $\mu = 1/2$

$$(a\ \exp(-c_y)+1)^{-1}\, Z_{c_y} - (a+1)^{-1}\, Z_0 = 2\,\beta_y - y,$$

where β is a Brownian motion. It follows, because $Z_t \to 0$ as $t \to \infty$, that

$$\int_0^\infty \Big(a\exp(B^{(1/2)}_s)+1\Big)^{-2} \mathbf{1}_{\{B^{(1/2)}_s<0\}} ds \overset{\text{(d)}}{=} \inf\{y:\ \beta^{(1/2)}_y = \xi\},$$

where $\xi = Z_0/2(a+1)$ is exponentially distributed with parameter $(a+1)$.

□

Remark 3.6 The formula (16) in Remark 2.2 gives

$$\int_0^{H_p(B^{(1/2)})} \frac{ds}{(a\exp(B^{(1/2)}_s)+1)^2} = H_q(\beta^{(1/2)}),$$

where $p > 0$, and $q = \log\Big((a+1)/(a+\exp(-p))\Big)$.

We proceed by computing the joint Laplace transform of the functionals appearing in (31) and (32). This Laplace transform is also given in [39] but here we give a new derivation which from our point of view has independent interest. Define

$$\Delta^{(\pm)}_a := \int_0^\infty \frac{\mathbf{1}_{\{B^{(1/2)}_s \in \mathbf{R}_\pm\}}}{(a\exp(B^{(1/2)}_s)+1)^2}\, ds.$$

Proposition 3.7 *For non-negative k, K and c*

$$F(k,c,K) := \mathbf{E}_0\Big(\exp\Big(-k\,\Delta_a^{(+)} - c\,L^0_\infty(B^{(1/2)}) - K\,\Delta_a^{(-)}\Big)\Big) \tag{35}$$
$$= \frac{\sqrt{8k+1}\,\exp(\frac{r}{2})}{\sqrt{8k+1}\cosh(\frac{r}{2}\sqrt{8k+1}) + (2c(a+1)+\sqrt{8K+1})\sinh(\frac{r}{2}\sqrt{8k+1})}.$$

where $r = \log((a+1)/a)$. In particular, with the notation as in Proposition 3.5,

$$\begin{aligned} F(k,0,k) &= \mathbf{E}_0\Big(\exp\Big(-k\int_0^\infty (a\exp(B_s^{(1/2)})+1)^{-2}\,ds\Big)\Big) \\ &= \mathbf{E}_0\Big(\exp\Big(-k\,H_r(\beta^{(1/2)})\Big)\Big) \\ &= \exp\Big(-\frac{r}{2}(\sqrt{8k+1}-1)\Big), \end{aligned}$$

$$\begin{aligned} F(0,0,K) &= \mathbf{E}_0\Big(\exp\Big(-K\int_0^\infty \frac{\mathbf{1}_{\{B_s^{(1/2)}<0\}}}{(a\exp(B_s^{(1/2)})+1)^2}\,ds\Big)\Big) \\ &= \mathbf{E}_0\Big(\exp\Big(-K\,H_\lambda(\beta^{(1/2)})\Big)\Big) \\ &= \frac{1+a}{a+\frac{1}{2}+\sqrt{2K+\frac{1}{4}}}, \end{aligned}$$

and

$$\begin{aligned} F(k,0,0) &= \mathbf{E}_0\Big(\exp\Big(-k\int_0^\infty \frac{\mathbf{1}_{\{B_s^{(1/2)}>0\}}}{(a\exp(B_s^{(1/2)})+1)^2}\,ds\Big)\Big) \\ &= \mathbf{E}_0\Big(\exp\Big(-k\,H_r(\tilde\beta^{(1/2)})\Big)\Big) \\ &= \frac{\sqrt{8k+1}\,\exp(\frac{r}{2})}{\sqrt{8k+1}\cosh(\frac{r}{2}\sqrt{8k+1}) + \sinh(\frac{r}{2}\sqrt{8k+1})}. \end{aligned}$$

Moreover, $\Delta_a^{(+)}$ and $\Delta_a^{(-)}$ are conditionally independent given $L^0_\infty(B^{(1/2)})$.

Proof We begin as in [39] and express the functionals $\Delta_a^{(\pm)}$ in terms of a Brownian motion without drift. Firstly, notice that

$$\Delta_a^{(+)} \overset{(d)}{=} \int_0^\infty \frac{\exp(2\,B_s^{(-1/2)})\,\mathbf{1}_{\{B_s^{(-1/2)}<0\}}}{(a+\exp(B_s^{(-1/2)}))^2}\,ds,$$

and

$$\Delta_a^{(-)} \overset{(\mathrm{d})}{=} \int_0^\infty \frac{\exp(2\, B_s^{(-1/2)})\, \mathbf{1}_{\{B_s^{(-1/2)}>0\}}}{(a+\exp(B_s^{(-1/2)}))^2}\, ds.$$

We apply the Lamperti transformation (28) with $\mu = -1/2$ and $a = 1$ on the left hand sides of these identities. Recalling that $R^{(1)}$ is, in fact, a Brownian motion killed when it hits 0 we obtain

$$\Delta_a^{(+)} \overset{(\mathrm{d})}{=} \int_0^{H_0(B')} \frac{\mathbf{1}_{\{B'_s<1\}}}{(a+B'_s)^2}\, ds, \quad \Delta_a^{(-)} \overset{(\mathrm{d})}{=} \int_0^{H_0(B')} \frac{\mathbf{1}_{\{B'_s>1\}}}{(a+B'_s)^2}\, ds,$$

where B' is a Brownian motion started at 1. Instead of working with a Brownian motion B' starting at 1 we introduce a "new" Brownian motion $B = 1 - B'$ starting at 0. Using spatial homogeneity of Brownian motion we obtain

$$\int_0^{H_0(B')} \frac{\mathbf{1}_{\{B'_s<1\}}}{(a+B'_s)^2}\, ds \overset{(\mathrm{d})}{=} \int_0^{H_1(B)} \frac{\mathbf{1}_{\{B_s>0\}}}{(a+1-B_s)^2}\, ds =: \Delta^+,$$

and

$$\int_0^{H_0(B')} \frac{\mathbf{1}_{\{B'_s>1\}}}{(a+B'_s)^2}\, ds \overset{(\mathrm{d})}{=} \int_0^{H_1(B)} \frac{\mathbf{1}_{\{B_s<0\}}}{(a+1-B_s)^2}\, ds =: \Delta^-.$$

By the occupation time formula

$$\begin{aligned}
&(\,\Delta_a^{(+)},\ L^0_\infty(B^{(1/2)}),\ \Delta_a^{(-)}\,) \\
&\qquad \overset{(\mathrm{d})}{=} (\,\Delta^+,\ L^0_{H_1}(B),\ \Delta^-\,) \\
&\qquad = \Big(\int_0^1 \frac{L^{1-y}_{H_1}(B)}{(a+y)^2}\, dy\ ,\ L^0_{H_1}(B)\ ,\ \int_1^\infty \frac{L^{1-y}_{H_1}(B)}{(a+y)^2}\, dy \Big),
\end{aligned}$$

where $L^{1-y}_{H_1}(B)$ is the local time of B at $1-y$ up to $H_1(B)$. From the proof of (32) in Proposition 3.5 we know

$$\begin{aligned}
&\mathbf{E}_0\Big(\exp\Big(-K\,\Delta^-\Big) \mid L^0_{H_1}(B) = u\Big) \\
&\qquad = \mathbf{E}_0\Big(\exp\Big(-K\,\Delta_a^{(-)}\Big) \mid L^0_\infty(B^{(1/2)}) = u\Big) \\
&\qquad = \exp\Big(-\frac{u}{4(1+a)}(\sqrt{8\,K+1}-1)\Big).
\end{aligned}$$

From the Ray–Knight Theorem 5.1 it now follows

$$\begin{aligned}
&\mathbf{E}_0\Big(\exp\Big(-k\,\Delta_a^{(+)} - c\,L_\infty^0(B^{(1/2)}) - K\,\Delta_a^{(-)}\Big)\Big)\\
&\quad= \mathbf{E}_0\Big(\exp\Big(-k\int_0^1 \frac{X_s^{(2)}}{(a+s)^2}\,ds\, - c\,X_1^{(2)} - \frac{X_1^{(2)}}{4\,\alpha}(\sqrt{8K+1}-1)\Big)\Big)\\
&\quad= \mathbf{E}_0\Big(\exp\Big(-k\int_0^1 \frac{X_s^{(2)}}{(a+s)^2}\,ds\, - \gamma\,X_1^{(2)}\Big)\Big), \qquad (36)
\end{aligned}$$

where $X^{(2)}$ denotes the 2-dimensional squared Bessel process started at 0 and

$$\gamma = c + \frac{1}{4\,(a+1)}(\sqrt{8K+1}-1). \qquad (37)$$

Next, recall from Pitman and Yor [35] or Revuz and Yor [36] Exercise 1.34 p. 453 that for a general squared Bessel process $X^{(\delta)}$, $\delta \geq 2$, and for any positive Radon measure m on $[0,\infty)$, we have for $x \geq 0$ and $t > 0$

$$\mathbf{E}_x\Big(\exp\Big(-\frac{1}{2}\int_{(0,t]} X_s^{(\delta)}\,m(ds)\Big)\Big) = \phi(t)^{\delta/2}\exp\Big(\frac{x}{2}\,\phi'(0)\Big)$$

where $v \mapsto \phi(v)$ is a unique positive, continuous and non-increasing function satisfying for $0 \leq u \leq v \leq t$

$$\phi'(v) - \phi'(u) = \int_{(u,v]} \phi(s)\,m(ds), \qquad \phi(0) = 1, \qquad (38)$$

and which is a constant for $v \geq t$. By ϕ' we mean the right hand side derivative of ϕ. Notice from (38) that ϕ is convex in $(0,\infty)$. Choosing

$$m(A) = \int_A \frac{2k\,ds}{(a+s)^2} + 2\gamma\,\varepsilon_{\{1\}}(A),$$

where A is a Borel set in $(0,\infty)$ and $\varepsilon_{\{1\}}$ is the Dirac measure at 1, yields

$$\begin{aligned}
&\mathbf{E}_0\Big(\exp\Big(-\frac{1}{2}\int_{(0,1]} X_s^{(2)}\,m(ds)\Big)\Big)\\
&\qquad= \mathbf{E}_0\Big(\exp\Big(-k\int_0^1 \frac{X_s^{(2)}}{(a+s)^2}\,ds\, - \gamma\,X_1^{(2)}\Big)\Big)\\
&\qquad= \phi(1). \qquad (39)
\end{aligned}$$

To find the function ϕ, we proceed in a similar manner as in [35] Example 1 p. 432. It follows from (38) that $\phi(v)$ for $v \le t$ is the (continuous) solution of

$$\phi''(v) = \frac{2k}{(a+v)^2}\,\phi(v), \tag{40}$$

satisfying the conditions (notice that $\phi'(1) = 0$)

$$\phi(0) = 1, \quad \phi'(1-) = -2\gamma\phi(1). \tag{41}$$

It is elementary to check that $y(v) = (a+v)^\alpha$ is a solution of (40) if and only if $\alpha(\alpha-1) = 2k$, that is

$$\alpha = \frac{1}{2}(1 \pm \sqrt{8k+1}) =: \alpha_\pm.$$

Introducing

$$y_+(v) := (1+\frac{v}{a})^{\alpha_+} \quad \text{and} \quad y_-(v) := (1+\frac{v}{a})^{\alpha_-},$$

our task is to find constants A and B such that

$$\phi(v) := A\,y_+(v) + B\,y_-(v),$$

fullfills (41). We skip the detailed computations and state only the result needed in (39):

$$\phi(1) = w\left(y_+'(1) + 2\gamma y_+(1) - (y_-'(1) + 2\gamma y_-(1))\right)^{-1},$$

where $w = \sqrt{8k+1}/a$ is the Wronskian. The desired formula (35) in Proposition 3.7 results now from (36) and (39) by recalling the definition of γ in (37) and substituting $r = \log((a+1)/a)$. □

Remark 3.8 It is seen from the proof above (take $\gamma = 0$) that for $a > 0$

$$\mathbf{E}_0\left(\exp\left(-k\int_0^1 \frac{X_s^{(\delta)}}{(a+s)^2}\,ds\right)\right) = \left(\frac{\sqrt{8k+1}}{a}\right)^{\delta/2} \left(y_+'(1) - y_-'(1)\right)^{-\delta/2}.$$

This formula is valid also for $0 < \delta < 2$ when the boundary point 0 is taken to be reflecting. In particular, $X^{(1)}$ is a Brownian motion squared. Similarly, for $a > 1$

$$\mathbf{E}_0 \left(\exp \left(-k \int_0^1 \frac{X_s^{(\delta)}}{(a-s)^2}\, ds \right) \right)$$
$$= \left(\frac{\sqrt{8k+1}}{a} \right)^{\delta/2} \left(x'_-(1) - x'_+(1) \right)^{-\delta/2}, \qquad (42)$$

where for $v < a$

$$x_+(v) := (1 - \frac{v}{a})^{\alpha_+} \quad \text{and} \quad x_-(v) := (1 - \frac{v}{a})^{\alpha_-}.$$

The formula (42) is derived in Mansuy [27] for squared Brownian motion using different techniques (but it is also indicated therein that the result can be obtained in the way presented above). See also Mansuy [26] for closely related results.

3.4 An identity due to Biane and Imhof

Next we consider the identity (5) (renumbered (43) below) found by Biane [2] and Imhof [19]. This identity is also observed in [38] in connection with a storage process. The distribution of the random variable H_λ featuring on the right hand side of (43) is in this context called the RBrownian motion-equilibrium-time-to-emptiness distribution, see Abate and Whitt [1].

We give two proofs of the Biane–Imhof identity (43). The first one is based on the Ray–Knight Theorem 5.2 and the random time change techniques. Given these tools the proof itself is very short. In fact, we repeat the idea of the proof of (32) in Proposition 3.5. The second proof is also based on random time changes but now we work via Tanaka's formula. This latter presentation is close to the one in [11] (see also Biane [2]), but we wish to give it anyway to demonstrate the connections between occupation and hitting times.

Proposition 3.9 *For $\mu > 0$*

$$\int_0^\infty \mathbf{1}_{\{B_s^{(\mu)} \leq 0\}}\, ds \quad \overset{(\mathrm{d})}{=} \quad H_\lambda(B^{(\mu)}), \qquad (43)$$

where $B^{(\mu)}$ is started at 0 and λ is an exponentially (with parameter 2μ) distributed random variable independent of $B^{(\mu)}$.

Proof 1 (based on the Ray–Knight Theorem 5.2.) We use the notation and the structure of the proof of (32) in Proposition 3.5. Firstly, by the occupation time formula and Theorem 5.2

$$\int_0^\infty \mathbf{1}_{\{B_s^{(\mu)}<0\}} ds = \int_{-\infty}^0 L_\infty^y(B^{(\mu)})\, dy \ \stackrel{\text{(d)}}{=} \ \int_0^\infty Z_y\, dy,$$

where Z satisfies

$$dZ_y = 2\sqrt{Z_y}\, dB_y - 2\mu Z_y\, dy. \tag{44}$$

The distribution of Z_0 is the distribution of $L_\infty^0(B^{(\mu)})$, i.e., the exponential distribution with parameter μ. For the random time change, introduce

$$C_y := \int_0^y Z_u\, du$$

and let c denote the inverse of C. It follows from (44)

$$Z_{c_y} - Z_0 = 2\,\beta_y - 2\mu\, y,$$

where β is a Brownian motion. Consequently, because $Z_y \to 0$ as $y \to \infty$ we obtain (43). □ *Proof 2 (based on the Tanaka formula).* Consider

$$(B_t^{(\mu)})^- = -\int_0^t \mathbf{1}_{\{B_s^{(\mu)}\le 0\}}\, dB_s^{(\mu)} + \frac{1}{2}\, L_t^0(B^{(\mu)}),$$

where $L^0(B^{(\mu)})$ is the local time of $B^{(\mu)}$ at 0 (with respect to the Lebesgue measure). Introduce

$$A_t^- := \int_0^t \mathbf{1}_{\{B_s^{(\mu)}\le 0\}}\, ds, \quad \text{and} \quad \alpha_t^- := \inf\{s :\ A_s^- > t\}.$$

Then letting $\ell_t^{(\mu)} = \frac{1}{2}\, L_t^0(B^{(\mu)})$

$$\begin{aligned}
(B_{\alpha_t^-}^{(\mu)})^- = -B_{\alpha_t^-}^{(\mu)} \ &= \ -\int_0^{\alpha_t^-} \mathbf{1}_{\{B_s^{(\mu)}\le 0\}}\, dB_s^{(\mu)} + \ell_{\alpha_t^-}^{(\mu)} \\
&= \ -\int_0^{\alpha_t^-} \mathbf{1}_{\{B_s^{(\mu)}\le 0\}}\, dB_s - \mu \int_0^{\alpha_t^-} \mathbf{1}_{\{B_s^{(\mu)}\le 0\}}\, ds + \ell_{\alpha_t^-}^{(\mu)} \\
&= \ -\int_0^{\alpha_t^-} \mathbf{1}_{\{B_s^{(\mu)}\le 0\}}\, dB_s - \mu\, t + \ell_{\alpha_t^-}^{(\mu)}.
\end{aligned}$$

It is a straightforward application of the result due to Dambis, and Dubins and Schwarz (see Revuz and Yor [36] p. 181) that the process given by

$$\beta_t := \int_0^{\alpha_t^-} \mathbf{1}_{\{B_s^{(\mu)} \leq 0\}} \, dB_s$$

is an $\mathcal{F}_{\alpha_t^-}$–Brownian motion. Notice that because $B^{(\mu)}_{\alpha_t^-} \leq 0$ we have for all $t \geq 0$

$$\ell^{(\mu)}_{\alpha_t^-} \geq \beta_t + \mu\, t. \tag{45}$$

Clearly, $A_t^- = A_{\Lambda_0}^-$ for $t \geq \Lambda_0 := \Lambda_0(B^{(\mu)}) := \sup\{t : \ B_t^{(\mu)} = 0\}$. Consequently, $\{B^{(\mu)}_{\alpha_t^-} : \ t \geq 0\}$ is defined only for $t < A_{\Lambda_0}^-$ and

$$\lim_{t \to A_{\Lambda_0}^-} B^{(\mu)}_{\alpha_t^-} = B^{(\mu)}_{\Lambda_0} = 0.$$

Because α^- is the inverse of A^- and $\ell^{(\mu)}$ does not increase after Λ_0 we obtain

$$0 = -\beta_{A_{\Lambda_0}^-} - \mu A_{\Lambda_0}^- + \ell^{(\mu)}_{\Lambda_0} = -\beta_{A_{\Lambda_0}^-} - \mu A_{\Lambda_0}^- + \ell^{(\mu)}_{\infty}.$$

From (45) it now follows that

$$A_\infty^- = A_{\Lambda_0}^- = \inf\{t : \ \beta_t + \mu t = \ell^{(\mu)}_\infty\}.$$

Because β is a Brownian motion and the $\mathbf{P}_0$–distribution of $\ell^{(\mu)}_\infty$ is exponential with parameter 2μ it remains to prove that β and $\ell^{(\mu)}_\infty$ are independent. To do this, let

$$A_t^+ := \int_0^t \mathbf{1}_{\{B_s^{(\mu)} \geq 0\}} \, ds, \quad \text{and} \quad \alpha_t^+ := \inf\{s : \ A_s^+ > t\},$$

and proceed as above to obtain

$$\begin{aligned} (B^{(\mu)}_{\alpha_t^+})^+ = B^{(\mu)}_{\alpha_t^+} &= \int_0^{\alpha_t^+} \mathbf{1}_{\{B_s^{(\mu)} \geq 0\}} \, dB_s^{(\mu)} + \ell^{(\mu)}_{\alpha_t^+} \\ &= \gamma_t + \mu t + \ell^{(\mu)}_{\alpha_t^+}, \end{aligned}$$

where $\{\gamma_t : \ t \geq 0\}$ is a Brownian motion. Because $B^{(\mu)}_{\alpha_t^+}$ is well defined and non-negative for all $t \geq 0$ we deduce from Skorokhod's reflection equation that

$$\ell^{(\mu)}_{\alpha_t^+} = \sup_{0 \leq s \leq t} \{-\gamma_s - \mu s\},$$

and letting $t \to \infty$ gives

$$\ell_\infty^{(\mu)} = \sup_{s \geq 0} \{-\gamma_s - \mu s\}.$$

From the extended form of Knight's theorem (see [36] p. 183) we know that β and γ are independent and, therefore, also $\ell_\infty^{(\mu)}$ and β are independent. $\square$

3.5 LeGall's identity

From a representation formula in LeGall [24] we can deduce the result in (46) below, see Donati-Martin and Yor [10] p. 1044 and 1052. In this section we prove (46) using time reversal argument and time changes. This nice application of the classical time reversal result was pointed out to us by Y. Hariya (in the context of (47)).

Proposition 3.10 *The following formula holds:*

$$\int_0^\infty \exp\left(-2\, R_s^{(3)}\right) ds \overset{\text{(d)}}{=} H_1(R^{(2)}), \tag{46}$$

where the Bessel processes are started from 0.

Proof For $x > 0$ let $\Lambda_x(R^{(3)}) := \sup\{t : \; R_t^{(3)} = x\}$. Then

$$\int_0^{\Lambda_x(R^{(3)})} \exp\left(-2\, R_s^{(3)}\right) ds \overset{\text{(d)}}{=} \int_0^{H_0(B)} \exp\left(-2\, B_s\right) ds,$$

where B is a standard Brownian motion started from x. Let for $t > 0$

$$A_t := \int_0^t \exp\left(-2\, B_s\right) ds$$

and, as usual, α_t is its inverse. By the Lamperti transformation (25), the process $\{Z_t : \; t \geq 0\}$ where $Z_t := \exp(-B_{\alpha_t})$ is a 2-dimensional Bessel process. Clearly,

$$Z_0 = \exp(-x) \quad \text{and} \quad 0 < Z_t < 1 \quad \forall\, t < A_{H_0}.$$

Consequently,

$$A_{H_0} \overset{\text{(d)}}{=} H_1(Z),$$

and letting $x \to \infty$ proves (46). $\square$

3.6 Hariya's identity

We learned the identity in the next proposition from Y. Hariya [18] and offer here a proof which differs from Hariya's proof and is in the spirit of the present paper.

Proposition 3.11 *The following identity holds:*

$$\int_0^\infty (1+R_s^{(3)})^{-2\gamma}\,ds \stackrel{(d)}{=} H_{1/(\gamma-1)}(R^{(\delta)}), \tag{47}$$

where $\gamma > 1$, $\delta = (2\gamma-1)/(\gamma-1)$ *and the Bessel processes are started from 0.*

Proof By the scaling property of Bessel processes and the occupation time formula the right hand side of (47) can be written as

$$\begin{aligned} H_{1/(\gamma-1)}(R^{(\delta)}) &\stackrel{(d)}{=} (\gamma-1)^{-2} H_1(R^{(\delta)}) \\ &\stackrel{(d)}{=} \frac{1}{(\gamma-1)^2}\int_0^1 L^y_{H_1}(R^{(\delta)})\,dy. \end{aligned}$$

Consequently, by Theorem 5.3 in Appendix (with the notations as therein), (47) is now equivalent with

$$\int_0^\infty \frac{Z_y^{(2)}}{(1+y)^{2\gamma}}\,dy \stackrel{(d)}{=} (\delta-2)\int_0^1 \frac{\widehat{Z}^{(2)}_{y^{\delta-2}}}{y^{\delta-3}}\,dy. \tag{48}$$

In particular, when $\gamma = 2$ (and $\delta = 3$) the identity (48) takes the simple form

$$\int_0^\infty \frac{Z_y^{(2)}}{(1+y)^4}\,dy \stackrel{(d)}{=} \int_0^1 \widehat{Z}^{(2)}_y\,dy. \tag{49}$$

To prove (47) via (48) we use the well known fact that

$$\{Z_s^{(2)} :\ s \geq 0\} \stackrel{(d)}{=} \{(1+s)^2 \widehat{Z}^{(2)}_{1/(1+s)} :\ s \geq 0\}. \tag{50}$$

Notice that (49) follows directly from (50). However, to obtain (48) we have to work little more. Substituting $x = y^{\delta-2}$ on the right hand side of (48) shows that (48) is equivalent with

$$\int_0^\infty \frac{Z_y^{(2)}}{(1+y)^{2\gamma}}\,dy \stackrel{(d)}{=} \int_0^1 x^{2(\gamma-2)}\,\widehat{Z}^{(2)}_x\,dx. \tag{51}$$

By the representation (50) the left hand side of (51) takes the form

$$\int_0^\infty \frac{Z_y^{(2)}}{(1+y)^{2\gamma}}\,dy \overset{\text{(d)}}{=} \int_0^\infty \frac{\widehat{Z}^{(2)}_{1/(1+y)}}{(1+y)^{2\gamma-2}}\,dy,$$

and substituting here $x = 1/(1+y)$ leads to the right hand side of (51), completing the proof. □

Remark 3.12 Consider the Ciesielski–Taylor identity (4):

$$\int_0^\infty \mathbf{1}_{\{R_s^{(\delta+2)}<1\}}\,ds \quad \overset{\text{(d)}}{=} \quad H_1(R^{(\delta)}), \tag{52}$$

where $R^{(\delta)}$ is a Bessel process of dimension $\delta > 0$ started at 0. It is possible to express (52) in alternative forms using Hariya's identity and some simple transformations. Indeed, by Theorem 5.2 (b) we rewrite (52) first in the form (see Yor [44] p. 52)

$$\frac{1}{\delta-2}\int_0^1 \frac{\widehat{Z}^{(2)}_{y^{\delta-2}}}{y^{\delta-3}}\,dy \overset{\text{(d)}}{=} \frac{1}{\delta}\int_0^1 \frac{Z^{(2)}_{y^\delta}}{y^{\delta-1}}\,dy. \tag{53}$$

Applying (48) on the left hand side of (53) yields

$$\frac{1}{(\delta-2)^2}\int_0^\infty \frac{Z_y^{(2)}}{(1+y)^{2\gamma}}\,dy \overset{\text{(d)}}{=} \frac{1}{\delta}\int_0^1 \frac{Z^{(2)}_{y^\delta}}{y^{\delta-1}}\,dy. \tag{54}$$

Making the change of variables $x = y^\delta$ on the right hand side of (54) and recalling that $\gamma = (\delta-1)/(\delta-2)$ leads to

$$\frac{1}{(\delta-2)^2}\int_0^\infty \frac{Z_y^{(2)}}{(1+y)^{2(\delta-1)/(\delta-2)}}\,dy \overset{\text{(d)}}{=} \frac{1}{\delta^2}\int_0^1 \frac{Z_x^{(2)}}{x^{2(\delta-1)/\delta}}\,dx. \tag{55}$$

Further, substituting on the right hand side of (55) $y = 1/x$ and using the time inversion property of Bessel processes we obtain

$$\begin{aligned}
\frac{1}{(\delta-2)^2}\int_0^\infty & \frac{Z_y^{(2)}}{(1+y)^{2(\delta-1)/(\delta-2)}}\,dy \\
&\overset{\text{(d)}}{=} \frac{1}{\delta^2}\int_1^\infty y^{2/\delta}\; Z^{(2)}_{1/y}\,dy \\
&\overset{\text{(d)}}{=} \frac{1}{\delta^2}\int_1^\infty \frac{Z_z^{(2)}}{z^{2(\delta-1)/\delta}}\,dz \\
&\overset{\text{(d)}}{=} \frac{1}{\delta^2}\int_0^\infty \frac{Y_u^{(2)}}{(1+u)^{2(\delta-1)/\delta}}\,du,
\end{aligned}$$

where $Y^{(2)}$ denotes the process $Z^{(2)}$ started with an exponential distribution with parameter $1/2$ (which is the distribution of $Z_1^{(2)}$ when started from 0).

4 Feynman-Kac approach to perpetual integral functionals

In this section we show how the Feynman-Kac formula can be used to find the Laplace transform of a perpetual integral functional $I_\infty(f)$ where f satisfies the integrability condition (13):

$$\int^\infty f(y)dy < \infty.$$

We remark also that in special cases one can find the law of a perpetual functional by limiting procedures but for a general characterization the problem has to be analyzed more carefully.

For treatments of Feynman-Kac formula, we refer to Durrett [14] and Karatzas and Shreve [22]. See also Jeanblanc, Pitman and Yor [25] for connections with excursion theory.

Consider for $\gamma > 0$ (and $\mu > 0$) the function

$$x \mapsto \Psi_\gamma(x) = \mathbf{E}_x\Big(\exp(-I_\infty(\gamma f))\Big) = \mathbf{E}_x\Big(\exp\Big(-\gamma\int_0^\infty f(B_s^{(\mu)})\,ds\Big)\Big).$$

By the simple Markov property it is seen that the process

$$\Big\{\Psi_\gamma(B_t^{(\mu)})\,\exp\Big(-\gamma\int_0^t f(B_s^{(\mu)})\,ds\Big)\,:\,t\geq 0\Big\}$$

is a bounded martingale.

Proposition 4.1 *The function Ψ_γ is non-decreasing and satisfies for all x and $t \geq 0$*

$$\Psi_\gamma(x) = \mathbf{E}_x\Big(\Psi_\gamma(B_t^{(\mu)})\;\exp\Big(-\gamma\int_0^t f(B_s^{(\mu)})\,ds\Big)\Big). \tag{56}$$

Moreover,

$$\lim_{x\to\infty}\Psi_\gamma(x) = 1. \tag{57}$$

Proof The formula (56) is immediate from the martingale property. Let $x < y$ and apply the optional stopping theorem to obtain

$$\begin{aligned}\Psi_\gamma(x) &= \mathbf{E}_x\Big(\Psi_\gamma(B^{(\mu)}_{H_y})\,\exp\Big(-\gamma\int_0^{H_y} f(B^{(\mu)}_s)\,ds\Big)\Big) \\ &\leq \Psi_\gamma(y),\end{aligned} \tag{58}$$

which shows that Ψ_γ is non-decreasing. Next notice that

$$\Psi_\gamma(B^{(\mu)}_t)\,\exp\Big(-\gamma\int_0^t f(B^{(\mu)}_s)\,ds\Big) = \mathbf{E}\Big(\exp\Big(-\gamma\int_0^\infty f(B^{(\mu)}_s)\,ds\Big)\,|\,\mathcal{F}_t\Big).$$

Consequently, by the martingale convergence theorem,

$$\Psi_\gamma(\infty)\,\exp\Big(-\gamma\int_0^\infty f(B^{(\mu)}_s)\,ds\Big) = \exp\Big(-\gamma\int_0^\infty f(B^{(\mu)}_s)\,ds\Big)$$

showing (57). □

Remark 4.2 Notice from (58) that for $x \leq y$

$$\mathbf{E}_x\Big(\exp\Big(-\gamma\int_0^{H_y} f(B^{(\mu)}_s)\,ds\Big)\Big) = \frac{\Psi_\gamma(x)}{\Psi_\gamma(y)}.$$

In many cases the function Ψ_γ can be found by solving a second order ODE. This is formulated in the following

Proposition 4.3 *Assume that f is piecewise continuous and satisfies the integrability condition (13). Then $x \mapsto \Psi_\gamma(x)$ is the unique positive, non-decreasing and continuously differentiable solution of the problem*

$$\begin{aligned}&\frac{1}{2}\,v''(x) + \mu\,v'(x) - \gamma\,f(x)\,v(x) = 0, \\ &\lim_{x\to\infty} v(x) = 1.\end{aligned} \tag{59}$$

Proof Notice that $B^{(\mu)}$ killed according to the additive functional

$$I_t(\gamma f) = \gamma\int_0^t f(B^{(\mu)}_s)\,ds$$

is a linear diffusion $B^\bullet$ (in the sense of Ito and McKean [20]), and its basic characteristics (speed measure, scale function and killing measure) can be determined explicitly (see [3] No. II.9 p. 17). From (56) it follows that Ψ_γ is an invariant function for $B^\bullet$. It is well known that invariant functions of a linear diffusion are continuous (see Dynkin [15] Vol. II p. 7), and differentiable when the scale function is differentiable (see Salminen [37] p. 93). From the representation theory of excessive functions we know that for $B^\bullet$ there exist two invariant functions, denoted φ_γ and ψ_γ and called *fundamental solutions* or *extreme invariant functions*, such that if h is an arbitrary invariant function then there exists constants $c_1 \geq 0$ and $c_2 \geq 0$ such that $h = c_1\,\psi_\gamma + c_2\,\varphi_\gamma$. Moreover, we have

$$\mathbf{P}_x(H_z(B^\bullet) < \infty) = \begin{cases} \dfrac{\varphi_\gamma(x)}{\varphi_\gamma(z)}, & x \geq z, \\ \dfrac{\psi_\gamma(x)}{\psi_\gamma(z)}, & x \leq z, \end{cases}.$$

From this representation it follows that φ and ψ solve (59) on the intervals of continuity of f. Next notice that, because $\mu > 0$,

$$\lim_{z\to-\infty} \mathbf{P}_x(H_z(B^\bullet) < \infty) = 0$$

giving $\varphi_\gamma(z) \to +\infty$ as $z \to -\infty$. Consequently, all invariant non-decreasing functions are multiples of ψ_γ. In particular, Ψ_γ is a multiple of ψ_γ and, hence, the condition (57) determines Ψ_γ uniquely. $\square$

Remark 4.4 At the points of discontinuity of f the function Ψ_γ usually fails to be two times differentiable.

Example 4.5 We compute the Laplace transform of the functional

$$\int_0^\infty (a + \exp(B_s^{(1/2)}))^{-2}\, ds$$

appearing in (8). Consider, for a moment, the case with general $\mu > 0$. Taking

$$f(x) = (a + \exp(x))^{-2}$$

in (59) gives us the equation

$$\frac{1}{2}\,v''(x) + \mu\, v'(x) - \gamma\,(a+\exp(x))^{-2}\,v(x) = 0.$$

Putting here $x = \ln y$ and $g(y) = v(\ln y)$ yields

$$\frac{1}{2}\,y^2\,g''(y) + (\mu + \frac{1}{2})\,y\,g'(y) - \gamma\ (a+y)^{-2}\,g(y) = 0, \tag{60}$$

which is, of course, the corresponding equation for geometric Brownian motion. By Kamke [21] 2.394 p. 497 this equation can be solved for $\mu = 1/2$ by making the substitution

$$\eta(\xi) = g(y), \quad \xi = \frac{\sqrt{2\gamma}}{a}\ln(\frac{y}{y+a}),$$

which transforms (60) to the following

$$\sqrt{2\gamma}\ \eta'' + a\ \eta' = \sqrt{2\gamma}\ \eta. \tag{61}$$

Letting $\beta := a/2\sqrt{2\gamma}$ the general solution of (61) can be written as

$$\eta(\xi) = A\ \exp\Big(-(\sqrt{1+\beta^2}+\beta)\xi\Big) + B\ \exp\Big((\sqrt{1+\beta^2}-\beta)\xi\Big).$$

Consequently, the increasing solution of (60) is

$$\begin{aligned}\psi(y) &= \exp\Big((\sqrt{1+\beta^2}-\beta)\frac{1}{2\beta}\ln(\frac{y}{a+y})\Big)\\ &= \Big(\frac{y}{a+y}\Big)^{2^{-1}(\sqrt{1+\beta^{-2}}-1)}.\end{aligned}$$

Notice that $\psi(\infty) = 1$, and it follows

$$\begin{aligned}&\mathbf{E}_x\Big(\exp\Big(-\gamma\int_0^\infty (a+\exp(B_s^{(1/2)}))^{-2}\,ds\Big)\Big)\\ &\qquad\qquad = \Big(\frac{\exp(x)}{a+\exp(x)}\Big)^{(2\,a)^{-1}\,(\sqrt{a^2+8\gamma}-a)}.\end{aligned} \tag{62}$$

For a geometric Brownian motion X with $X_0 = x > 0$ defined via

$$X_s = \exp(B_s^{(1/2)}), \quad B_0^{(1/2)} = x,$$

the formula (62) takes the form

$$\mathbf{E}_x\Big(\exp\Big(-\gamma\int_0^\infty \frac{1}{(a+X_s)^2}\,ds\Big)\Big) = \Big(\frac{x}{a+x}\Big)^{(2\,a)^{-1}\,(\sqrt{a^2+8\gamma}-a)}. \tag{63}$$

The identity in law in (8) can be deduced from (62) (or (63)). Notice also that substituting in (62) $x = 0$ and letting $a \to 0$ we obtain a special case of the identity (2):

$$\mathbf{E}_0\Big(\exp\Big(-\gamma\int_0^\infty e^{2B_s - s}\,ds\Big)\Big) = e^{-\sqrt{2\gamma}},$$

i.e.,

$$\int_0^\infty e^{2B_s - s}\,ds \stackrel{(\mathrm{d})}{=} \Big(2\,Z_{1/2}\Big)^{-1},$$

where $Z_{1/2}$ is a Γ–distributed r.v. with parameter $1/2$.

Using Propositions 4.1 and 4.3 we derive an interesting result due to Biane [2] which characterizes the law of a perpetual integral functional of $B^{(\mu)}$, $\mu > 0$, restricted on $\mathbf{R}_-$ in terms of the same but unrestricted functional of another diffusion stopped at the first hitting time. We remark that in [2] a more general situation (not only $B^{(\mu)}$) is considered. However, the main interest in [2] is focused on occupation times, the aim being to generalize the Ciesielski–Taylor identity (4). The result in our Proposition 4.6 is extracted from Remarque p. 295 in [2] and formulated for $B^{(\mu)}$.

Proposition 4.6 *Let f be a positive $\mathcal{C}^1$-function such that*

$$\int_{-\infty} f(y)\,\mathrm{e}^{-2\mu y}\,dy = \infty, \tag{64}$$

and X a diffusion with the generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \Big(\mu - \frac{1}{2}\,\frac{f'(x)}{f(x)}\Big)\,\frac{d}{dx}.$$

Then $H_0(X) < \infty$ a.s. if $X_0 < 0$, and, moreover,

$$I_\infty^-(f) := \int_0^\infty f(B_s^{(\mu)})\,\mathbf{1}_{\{B_s^{(\mu)}<0\}}\,ds \quad \stackrel{(\mathrm{d})}{=} \quad \int_0^{H_0(X)} f(X_s)\,ds$$

where X_0 is taken to be exponentially distributed on $(-\infty, 0)$ with parameter 2μ.

Proof Notice that the condition (64) means that the scale function S^X of X given by (cf. [3] II.9 p. 17)

$$S^X(x) = \int^x f(y)\,\mathrm{e}^{-2\mu y}\,dy$$

satisfies

$$\lim_{x\to-\infty} S^X(x) = -\infty.$$

This implies $H_0(X) < +\infty$ a.s. when $X_0 < 0$. From Proposition 4.3 we know

$$\Psi_\gamma(x) := \mathbf{E}_x\Big(\exp\Big(-\gamma I^-_\infty(f)\Big)\Big)$$

is the unique, non-decreasing function such that

$$\frac{1}{2}\,\Psi''(x) + \mu\,\Psi'(x) = \gamma\mathbf{1}_{(-\infty,0)}(x)\,f(x)\,\Psi(x) \tag{65}$$

and $\lim_{x\to+\infty}\Psi(x) = 1$. For $x > 0$ we clearly have

$$\Psi_\gamma(x) = \mathbf{P}_x(H_0(B^{(\mu)}) = +\infty) + \mathbf{P}_x(H_0(B^{(\mu)}) < +\infty)\,\Psi_\gamma(0),$$

and, hence, it is enough to compute $\Psi_\gamma(x)$ for $x \le 0$. For this, consider the equation

$$\frac{1}{2}\,u''(x) + \mu\,u'(x) = \gamma\,f(x)\,u(x). \tag{66}$$

Let ψ_γ and φ_γ denote the fundamental non-decreasing and non-increasing, respectively, solutions of (65), and, similarly, $\widehat{\psi}_\gamma$ and $\widehat{\varphi}_\gamma$ are the fundamental non-decreasing and non-increasing, respectively, solutions of (66). Notice that f does not have to satisfy the integrability condition (13). However, for (66), we can still argue as in the proof of Proposition 4.3 that all invariant non-decreasing functions are multiples of $\widehat{\psi}_\gamma$. Using continuity and differentiability requirements, ψ_γ can be expressed in terms of $\widehat{\psi}_\gamma$ as follows

$$\psi_\gamma(x) = \begin{cases} \widehat{\psi}_\gamma(x), & x \le 0, \\[2ex] S(x)\,\dfrac{\widehat{\psi}'_\gamma(0)}{S'(0)} + \widehat{\psi}_\gamma(0) - \dfrac{S(0)\,\widehat{\psi}'_\gamma(0)}{S'(0)}, & x \ge 0, \end{cases}$$

where $S(x) = -\exp(-2\mu\, x)$ is the scale function of $B^{(\mu)}$. Consequently, for $x \leq 0$

$$\Psi_\gamma(x) = \frac{\psi_\gamma(x)}{\psi_\gamma(+\infty)} = \frac{2\mu\, \widehat{\psi}_\gamma(x)}{2\mu\, \widehat{\psi}_\gamma(0) + \widehat{\psi}_\gamma'(0)},$$

and, in particular, for $x = 0$

$$\mathbf{E}_0\Big(\exp\Big(-\gamma \int_0^\infty f(B_s^{(\mu)})\, \mathbf{1}_{\{B_s^{(\mu)}<0\}}\, ds \Big)\Big) = \frac{2\mu\, \widehat{\psi}_\gamma(0)}{2\mu\, \widehat{\psi}_\gamma(0) + \widehat{\psi}_\gamma'(0)}. \tag{67}$$

To proceed, define for $y \leq 0$

$$\widetilde{\psi}_\gamma(y) := 2\mu\, \widehat{\psi}_\gamma(y) + \widehat{\psi}_\gamma'(y),$$

and notice that $\widetilde{\psi}_\gamma''$ exists and is continuous because $f \in \mathcal{C}^1$. Using the fact that $\widehat{\psi}_\gamma$ solves (66) it is straightforward to verify that $\widetilde{\psi}_\gamma' > 0$, i.e., $\widetilde{\psi}_\gamma$ is increasing, and that $\widetilde{\psi}_\gamma$ is a solution of the ODE

$$\frac{1}{2}u''(x) + \Big(\mu - \frac{1}{2}\,\frac{f'(x)}{f(x)}\Big)\, u'(x) = \gamma\, f(x)\, u(x), \quad x \leq 0.$$

By Itô's formula, the process

$$\Big\{ \widetilde{\psi}_\gamma(X_{t\wedge H_0(X)})\, \exp(-\gamma \int_0^{t\wedge H_0(X)} f(X_s)\, ds) :\ t \geq 0 \Big\}$$

is a martingale and, further, because it is bounded, we obtain for $X_0 = x < 0$ by the dominated convergence theorem

$$\mathbf{E}_x\Big(\exp\Big(-\gamma \int_0^{H_0(X)} f(X_s)\, ds \Big)\Big) = \frac{\widetilde{\psi}_\gamma(x)}{\widetilde{\psi}_\gamma(0)}. \tag{68}$$

Observe that

$$\int_{-\infty}^0 \widetilde{\psi}_\gamma(x)\, 2\mu\, \mathrm{e}^{2\mu x}\, dx = 2\mu \int_{-\infty}^0 (2\mu \widehat{\psi}_\gamma(x) + \widehat{\psi}_\gamma'(x))\, \mathrm{e}^{2\mu x}\, dx = 2\mu \widehat{\psi}_\gamma(0).$$

Consequently, if X_0 is exponentially distributed on $(-\infty, 0)$ with parameter 2μ, (67) and (68) lead to

$$\begin{aligned}\int_{-\infty}^0 2\mu\, \mathrm{e}^{2\mu x}\, \mathbf{E}_x\Big(\exp\Big(-\gamma \int_0^{H_0(X)} f(X_s)\, ds \Big)\Big)\, dx \\ = \mathbf{E}_0\Big(\exp\Big(-\gamma \int_0^\infty f(B_s^{(\mu)})\, \mathbf{1}_{\{B_s^{(\mu)}<0\}}\, ds \Big)\Big),\end{aligned}$$

as claimed. □

Example 4.7 Consider the functional

$$I := \int_0^\infty \exp(-2\, B_s^{(\mu)})\, \mathbf{1}_{\{B_s^{(\mu)}<0\}}\, ds.$$

Recall from (22), that there exists a Bessel process $R^{(2-2\mu)}$ started from 1 such that

$$I \;=\; \int_0^\infty \mathbf{1}_{\{R_s^{(2-2\mu)}>1\}}\, ds \quad \text{a.s.}$$

As an application of Proposition 4.6 we derive a new charaterization of the distribution of I. Taking $f(x) = \mathrm{e}^{-2x}$ it is seen that the diffusion X in Proposition 4.6 is a Brownian motion with drift $\mu+1$. Consequently,

$$I \;\overset{\text{(d)}}{=}\; \int_0^{H_0(B^{(\mu+1)})} \exp\left(-2\, B_s^{(\mu+1)}\right) ds,$$

where $B_0^{(\mu+1)}$ is exponentially distributed on $(-\infty, 0)$ with parameter 2μ. To develop this further, let $x > 0$ and assume that $B_0^{(\mu+1)} = -x < 0$. We have

$$\begin{aligned}
&\int_0^{H_0(B^{(\mu+1)})} \exp\left(-2\, B_s^{(\mu+1)}\right) ds \\
&\qquad\overset{\text{(d)}}{=}\; \int_0^{H_x(\hat{B}^{(\mu+1)})} \exp\left(-2\,(\hat{B}_s^{(\mu+1)} - x)\right) ds \\
&\qquad=\; \mathrm{e}^{2x} \int_0^{H_x(\hat{B}^{(\mu+1)})} \exp\left(-2\, \hat{B}_s^{(\mu+1)}\right) ds \\
&\qquad\overset{\text{(d)}}{=}\; \mathrm{e}^{2x}\, \inf\{t\,:\, R_t^{(-2\mu)} = \mathrm{e}^{-x}\},
\end{aligned}$$

where $\hat{B}^{(\mu+1)}$ is a Brownian motion with drift $\mu+1$ started from 0 and in the last step Remark 2.2 is applied. Using the scaling property of Bessel processes we obtain

$$\begin{aligned}
\inf\{t\,:\, R_t^{(-2\mu)} = \mathrm{e}^{-x}\} &= \inf\{t\,:\, \mathrm{e}^{x}\, R_t^{(-2\mu)} = 1\} \\
&= \inf\{\mathrm{e}^{-2x}\, t\,:\, \mathrm{e}^{x}\, R_{\mathrm{e}^{-2x}\,t}^{(-2\mu)} = 1\} \\
&\overset{\text{(d)}}{=}\; \mathrm{e}^{-2x}\, \inf\{t\,:\, \hat{R}_t^{(-2\mu)} = 1\},
\end{aligned}$$

where the Bessel process $\hat{R}^{(-2\mu)}$ is started from e^{x}. Consequently,

$$\int_0^\infty \exp(-2\, B_s^{(\mu)})\, \mathbf{1}_{\{B_s^{(\mu)}<0\}}\, ds. \;\overset{\text{(d)}}{=}\; H_1(\hat{R}^{(-2\mu)}), \tag{69}$$

where $\hat{R}_0^{(-2\mu)}$ is distributed as e^{ξ} with ξ exponentially distributed with parameter 2μ. Elementary computations show that

$$\mathbf{P}(\hat{R}_0^{(-2\mu)} > z) = z^{-2\mu}, \quad z \geq 1. \tag{70}$$

It is interesting to notice that the right hand side of (70) when extended to a measure on the whole of $\mathbf{R}_+$ can be viewed as the speed measure of $\hat{R}^{(-2\mu)}$ (see, e.g., [3] A1.21 p. 133).

5 Appendix on Ray–Knight theorems

For an easy reference, we recall here the Ray–Knight theorems used in this paper (see Yor [44] and [3] for more complete statements).

Theorem 5.1 *Let B be a standard Brownian motion started from 0 and $L^y_{H_1(B)}(B)$ its local time (with respect to the Lebesgue measure) at level $y \leq 1$ up to $H_1(B)$. Then the local time process $\{L^{1-y}_{H_1}(B) : \ y \geq 0\}$ is a solution of the SDE*

$$X_y = 2\int_0^y \sqrt{X_s}\, d\beta_s + 2(\, y \wedge 1\,),$$

in other words,

(i) $\{L^{1-y}_{H_1(B)}(B) : \ 0 \leq y \leq 1\}$ is a 2-dimensional squared Bessel process starting from 0,

(ii) $\{L^{1-y}_{H_1(B)}(B) : \ y \geq 1\}$ is a 0-dimensional squared Bessel process with the starting value $L^0_{H_1}(B)$ obtained from (i).

Theorem 5.2 *Assume that $B^{(\mu)}_0 = 0$ and let $L^y_t(B^{(\mu)})$ be the local time of $B^{(\mu)}$ at level y (with respect to the Lebesgue measure) up to time t. Define the total local time of $B^{(\mu)}$ at level y via*

$$L^y_\infty(B^{(\mu)}) := \lim_{t\to\infty} L^y_t(B^{(\mu)}).$$

Then

$$\{L^{-y}_\infty(B^{(\mu)}) : \ y \geq 0\} \overset{(\mathrm{d})}{=} \{Z^{(0,2\mu)}_y : \ y \geq 0\}, \tag{71}$$

and

$$\{L^y_\infty(B^{(\mu)}) :\ y \geq 0\} \overset{\text{(d)}}{=} \{Z^{(2,2\mu)}_y :\ y \geq 0\}, \tag{72}$$

where $Z^{(\delta,2\mu)}, \delta = 0, 2$, *are solutions of the SDE*

$$dX_t = 2\sqrt{X_t}\, dB_t + (\delta - 2\mu X_t)\, dt,$$

respectively, with the initial value X_0 *exponentially distributed with parameter* μ. *In fact, the identities (71) and (72) hold jointly, with* $Z^{(0,2\mu)}_0 = Z^{(2,2\mu)}_0$ *but otherwise* $Z^{(0,2\mu)}$ *and* $Z^{(2,2\mu)}$ *are independent.*

Our final Ray–Knight theorem is for Bessel processes. The first part is formulated only for 3-dimensional Bessel process, and in the second part we take the dimension parameter $\delta > 2$. Let $L^y_\infty(R^{(\delta)})$ denote the total local time at y of the Bessel process $R^{(\delta)}$ (taken with respect to the Lebesgue measure).

Theorem 5.3 a: *Assume that* $R^{(3)}$ *is started at 0. Then*

$$\{L^y_\infty(R^{(3)}) :\ y \geq 0\} \overset{\text{(d)}}{=} \{Z^{(2)}_y :\ y \geq 0\},$$

where $Z^{(2)}$ *denotes the squared Bessel process of dimension 2, started from 0, i.e.,* $Z^{(2)}$ *satisfies the SDE*

$$dX_y = 2\sqrt{X_y}\, dB_y + 2\, dy.$$

b: *Assume that* $\delta > 2$ *and* $R^{(\delta)}_0 = 0$. *Then*

$$\{L^y_{H_1}(R^{(\delta)}) :\ 0 \leq y \leq 1\} \overset{\text{(d)}}{=} \left\{ \frac{1}{(\delta-2)y^{\delta-3}}\ \widehat{Z}^{(2)}_{y^{\delta-2}} :\ 0 \leq y \leq 1 \right\}$$

where $\widehat{Z}^{(2)}$ *denotes the 2-dimensional squared Bessel bridge (from 0 to 0 and of length 1).*

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